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Mixed-region collapse in a stratified fluid

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A simplified theoretical model of a collapsing mixed region in a stably stratified fluid, as studied experimentally by Wu (1969), is presented. Unlike previous work the model describes the mixed region itself in the 'principal stage' of collapse. An inertia–buoyancy balance assumption is invoked, resulting in good quantitative agreement with Wu's results.

1. Background

When an object moves horizontally at speed through a stably stratified fluid it generates a turbulent wake, the longitudinal variation of which is small compared with its transverse variation (Schooley & Stewart 1963). To model the subsequent almost two-dimensional lateral spread, Wu (1969) conducted laboratory experiments on the gravitational collapse of a homogeneous circular cylinder of water at its equilibrium level in a salt-stratified environment.

The experiments of Schooley & Stewart (1963) showed that the mixed-region collapse is accompanied by excitation of internal waves in the stratified environment and this aspect of the problem has received considerable attention from Wu (1969), Mei (1969) and later writers.

A proper mathematical study of the mixed-region collapse itself is much more difficult than a study of the internal waves generated in the far field by the collapse because of the essential nonlinearity of the former problem. Thus Mei (1969), neglecting viscosity, considered the stratified fluid to be in hydrostatic equilibrium and used the long-wave approximation to the behaviour of the well-mixed region. He obtained only order-of-magnitude agreement with Wu's results however, and even then only for small times when the long-wave approximation is invalid. Padmanabhan *et al.* (1970) also considered the stratified environment to deform hydrostatically, ignoring any motions in that region. They solved numerically the inviscid problem with the potential function satisfying Laplace's equation inside the well-mixed section and a hydrostatic pressure approximation to Bernoulli's equation on its interface. The solution was in only fair agreement with Wu's results, probably owing in part to an error in the approximation to the interface condition used.

Wessel (1969), Young & Hirt (1972) and Orlanski & Ross (1973) have all performed numerical experiments on the collapse of a well-mixed region in a stratified environment, including the interaction of the two regions. With varying degrees of sophistication they each obtained fair agreement with the results of Wu (1969).

It seems that the only theoretical work which considers the reaction of the environment on the mixed region is the linearized treatment of Hartman & Lewis (1972). However their analysis is valid only for a partially mixed region: that is, the deviation from the mean environmental stratification in the initially circular collapsing region must be small. Viscous effects are not included, so the analysis becomes invalid for large times, breaking down first at the interface of the deforming, partially mixed region. Even so it gives a good idea of the behaviour of a fully mixed region for small times, when the omission of viscous effects can be justified. Hartman & Lewis found that the horizontal and vertical velocities for $r < a_0$ (the initial radius of the partially mixed region) behave like

$$u = \frac{2\epsilon x}{\gamma t} J_2(Nt), \quad w = -\frac{2\epsilon z}{\gamma t} J_2(Nt), \tag{1.1}$$

where ϵ ($\ll \gamma$) is the initial disturbance inside $r = a_0$ to the environmental density gradient $\rho_0 \gamma$ ($= \rho_0 N^2/g$; see figure 1) and J_2 is the second-order Bessel function of the first kind. Thus the fluid particles inside $r = a_0$ initially move on right hyperbolas, overshooting their ultimate positions on the first pass. The initially circular section deforms into an ellipse with oscillations around this shape. It must be because of such behaviour in the fully mixed problem (cf. Wu 1969, figure 2(*b*), stage 2) that internal waves are generated so readily.

To simulate accurately the interaction of the mixed region and environment requires complex numerical procedures such as the marker-and-cell technique used by Young & Hirt (1972). However, with the above comments in mind, the following approximate model of the two-dimensional gravitational collapse of an initially circular section of inviscid, non-diffusive, homogeneous fluid at its equilibrium level in a linearly stratified Boussinesq fluid is considered. No attempt is made to model the initial unsteady behaviour. This part of the collapse, occupying only one or two buoyancy periods at the most, is one of rapid adjustment from a motion with constant acceleration (inherent in the fluid motion starting from rest; see Mei 1969) to a decelerating motion. Rather, it is the 'principal stage' of collapse (Wu 1969) extending over $3 \leqslant Nt \leqslant 25$ which is considered, but even here the interaction of internal waves with the collapsing region is only implicitly assumed.

2. Simplified inviscid model of mixed-region collapse

Consider the collapse of the initially circular mixed region (figure 1). If the pressure were uniform on the surface S of the section, Longuet-Higgins (1972) has shown that subsequent deformation of S would be as an ellipse. In a gravitationally stratified environment the pressure on S is of course not uniform, but even so Hartman & Lewis (1972) have shown that S deforms as an ellipse for the partially mixed problem at least for small times. Mei's (1969) analysis, valid for large times, showed that an elliptic section continues to deform as an ellipse.

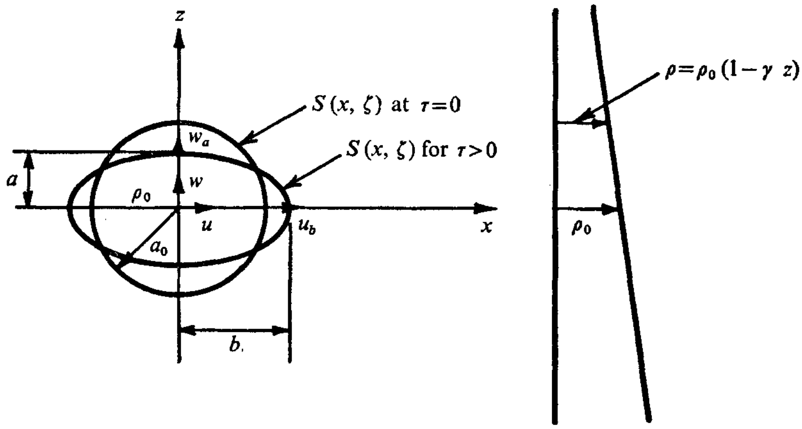

FIGURE 1. Definition sketch of the model of mixed-region collapse.

It is assumed here that the initially circular mixed region, which deforms without rotation, maintains an elliptic shape at all times. Thus, with b and a the major and minor axes of the elliptic interface $S(x, z = \zeta, t)$, as indicated in figure 1, S is defined by

$$x^2/b^2 + \zeta^2/a^2 = 1. \tag{2.1}$$

The fluid inside S is irrotational so there the velocity potential ϕ satisfies

$$\nabla^2\phi = 0 \quad \text{with} \quad u = \partial\phi/\partial x, \quad w = \partial\phi/\partial z. \tag{2.2}$$

Then, if the lengths of the axes of the ellipse are changing at the rates $\dot{b} = u_b$ and $\dot{a} = w_a$, the velocity potential satisfying (2.1) and (2.2) may be expressed as (Lamb 1932, §110)

$$\phi = \frac{1}{2}\left(\frac{\dot{b}}{b}x^2 + \frac{\dot{a}}{a}z^2\right), \tag{2.3}$$

where $\dot{b}/b + \dot{a}/a = 0$. This relation merely expresses conservation of area in S:

$$ab = a_0^2, \tag{2.4}$$

where a_0 is the radius of the initial circular section. It follows from (2.3) that inside S

$$u = Ax, \quad w = -Az. \tag{2.5}$$

$A \equiv \dot{b}/b$ is a function of time only. The instantaneous streamlines are rectangular hyperbolas.

Wu (1969) demonstrated that in the 'principal stage' of collapse the motion obeys internal Froude number scaling. It is postulated here that in the 'principal stage' the collapse happens so slowly that an instantaneous balance is maintained between horizontal inertia forces and buoyancy forces so far as the motion of the mixed region is concerned. Evidently just enough of the potential energy lost by the collapsing region goes into radiated 'internal waves' and (in practice) dissipation along its boundaries for the balance to hold. For the mixed region, then, a local internal Froude number is said to be time invariant in the 'principal stage':

$$Fr \equiv u/N\zeta = O(1), \quad \text{independent of time.} \tag{2.6}$$

N is the ambient buoyancy frequency $(g\gamma)^{\frac{1}{2}}$. Fr is a function of x/b only and varies smoothly from a value of zero at $x/b = 0$ to infinity at $x/b = 1$. An internal Froude number $\overline{Fr}$ characteristic of the mixed region as a whole is obtained by averaging over the section S:

$$\overline{Fr} = \int_0^1 Fr\, d(x/b).$$

By (2.1), (2.2) and (2.5),

$$\overline{Fr} = \int_0^1 \frac{Ab}{Na} \frac{x/b}{(1-(x/b)^2)^{\frac{1}{2}}} d(x/b)$$

$$= Ab/Na, \tag{2.7}$$

i.e.

$$\overline{Fr} = u_b/Na, \quad \text{a constant.} \tag{2.8}$$

u_b is the horizontal nose velocity of the mixed region at $x = b$. Note also that $\overline{Fr} = Fr(x/b)$ when $x/b = 1/\surd 2$. $\overline{Fr}$ is of the same form as that obtained for the continuous lateral intrusion of fluid into a stratified environment (Manins 1976). This is because much the same dynamics are involved, but of course the geometry and kinematics are quite different.

The behaviour of the section may easily be obtained: from (2.8) and (2.4)

$$\left.\begin{aligned} b &= a_0(1+2\overline{Fr}\,Nt)^{\frac{1}{2}}, \\ u_b &= \overline{Fr}\,Na_0/(1+2\,\overline{Fr}\,Nt)^{\frac{1}{2}}. \end{aligned}\right\} \tag{2.9}$$

so

From (2.4) and (2.9)

$$\left.\begin{aligned} a &= a_0/(1+2\overline{Fr}\,Nt)^{\frac{1}{2}}, \\ w_a &= -\overline{Fr}\,Na_0/(1+2\overline{Fr}Nt)^{\frac{3}{2}}, \end{aligned}\right\} \tag{2.10}$$

so

where $u_b = \dot{b}$ and $w_a = \dot{a}$. The initial conditions used in (2.9) and (2.10), viz. $b = a = a_0$ at $t = 0$, while not being strictly correct since the model holds only in the 'principal stage' of collapse, are imposed to close the equations with only $\overline{Fr}$ to be determined. Because the 'initial stage' of collapse represents only a small part of the whole range covered, discrepancies between the model and Wu's experiment due to the choice of initial conditions in (2.9*a*) and (2.10*a*) are negligible in a plot like figure 2.

Figure 2 compares (2.9*a*) for several values of $\overline{Fr}$ with Wu's (1969) results and also with the MAC calculations of Young & Hirt (1972). Wu's (1969, table 1) upper tangent points, plotted in figure 2 and representing the observed approximate changeover times from the 'principal stage' to the 'final stage' where apparently viscous forces predominate over inertia forces, are strangely at variance with Wu's (1969, table 2) 'best-fit' curve. It may be seen that to an approximation well within Wu's experimental uncertainties

$$\overline{Fr} = \tfrac{5}{8} \tag{2.11}$$

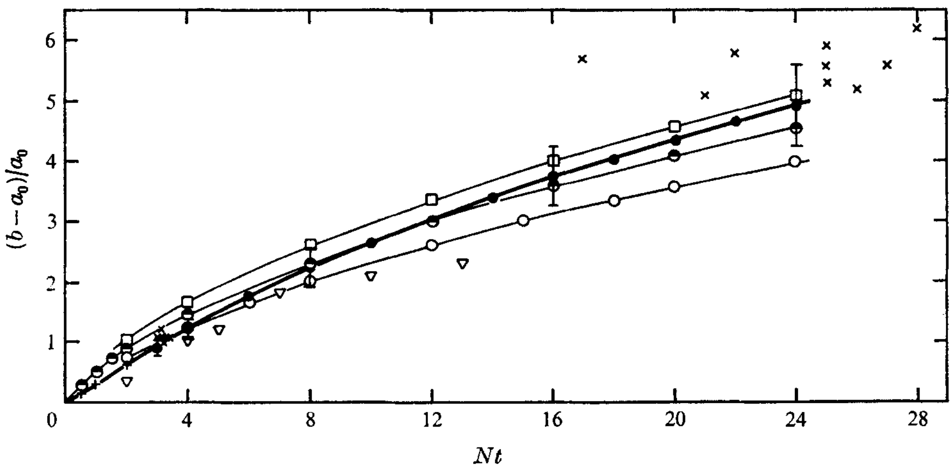

FIGURE 2. Plot of lateral extent of the mixed region as a function of time. —●—, Wu's (1969) 'best-fit' curve with maximum deviations observed indicated by vertical error bars; ×, Wu's (1969) tangent points; ▽, data points from Young & Hirt (1972). Present model: —○—, $\overline{Fr} = \frac{1}{2}$; —◓—, $\overline{Fr} = \frac{5}{8}$; —□—, $\overline{Fr} = \frac{3}{4}$.

is a good choice for the model. The agreement is not particularly sensitive to $\overline{Fr}$, which is $O(1)$ as expected. Now in the range $3 < Nt < 25$ Wu found that

$$b/a_0 = 1{\cdot}03(\pm 0{\cdot}05)\,(Nt)^{0{\cdot}55(\pm 0{\cdot}02)} \tag{2.12}$$

best fitted his results. For large Nt, (2.9a) reduces to

$$b/a_0 \sim (\tfrac{5}{4}Nt)^{\frac{1}{2}} = 1{\cdot}12(Nt)^{0{\cdot}50}. \tag{2.13}$$

The agreement between (2.12) and (2.13) is excellent, as is also evident from figure 2. The implication is that the collapse process is in a 'constant Froude number regime' for a major part of the time.

An estimate of the time of onset of the 'constant Froude number regime' may be made as follows. For an elliptic mixed region and hydrostatically deforming environment, the potential energy of the region at time t is

$$\mathrm{PE}(t) = \tfrac{1}{4}\pi\rho_0 N^2 a^3 b. \tag{2.14}$$

The kinetic energy for small times may be found by approximating the flow in the environment as potential. Then

$$\mathrm{KE}(t) = \tfrac{1}{4}\pi\rho_0 C(t)\,A^2 ab(a^2+b^2), \tag{2.15}$$

where $2C(t)$ is the inertia coefficient (Lamb 1932, §114) for the ellipse. $C(0)$ is unity and $C(t)$ decreases as the ellipticity for small times. At any time $t = \tau$ then

$$\mathrm{PE}(0) - \mathrm{PE}(\tau) = \mathrm{KE}(\tau) + \text{increase in energy of environment due to isopycnal distortions.} \tag{2.16}$$

For small times, $N\tau \lesssim 3$ say, the energy transferred to isopycnal distortions will be small. An estimated upper bound on the energy in internal waves observed

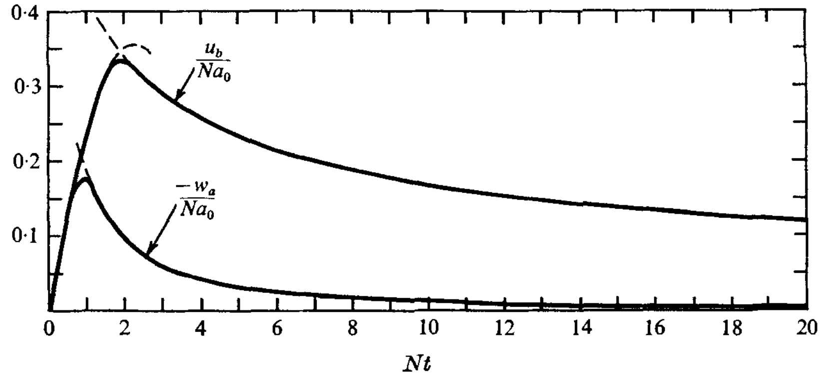

FIGURE 3. Velocities of the mixed region as a function of time according to the model. For $Nt \gtrsim 1{\cdot}3$, $-u_b/w_a = 1+2\overline{Fr}\,Nt$.

by Wu (1969, figure 5) at $N\tau = 3$ is less than 30 % of KE(τ). This energy may be thought of as being accounted for by a modified $C(t)$. Because the estimated transition time [see below, equation (2.20)] behaves like $C(t)^{-\frac{1}{8}}$ for Nt small, a deviation from unity of $C(t)$ will have a very small effect.

From (2.16) an expression for A valid for energy-conserving elliptic deformation is obtained:

$$A^2 = \frac{N^2}{C^2}\left(\frac{a}{a_0}\right)^2 \frac{1-(a/a_0)^2}{1+(a/a_0)^4}. \tag{2.17}$$

In the constant Froude number regime, from (2.8), (2.5a) and (2.4)

$$A^2 = \overline{Fr}^2 N^2 (a/a_0)^4. \tag{2.18}$$

The A's given by relations (2.17) and (2.18) will be approximately equal in the transition from energy-conserving flow to a balance flow characterized by (2.8). With $\overline{Fr}$ given by (2.11) it follows that there

$$a/a_0 \simeq 0{\cdot}80 \quad \text{for} \quad C = 1. \tag{2.19}$$

An indication of the time which elapses before (2.19) is satisfied may be obtained by formally setting $\epsilon = \gamma$ and integrating (1.1b). Then $a/a_0 = 2J_1(Nt)/(Nt)$ and from (2.19)

$$Nt \simeq 1{\cdot}3 \quad \text{at transition.} \tag{2.20}$$

Figure 3 illustrates the calculated behaviour of the mixed-region velocities u_b and w_a. For $Nt > 1$ equations (2.9b) and (2.10b) are plotted while for $Nt < 1$ the behaviour is approximated from (1.1). The segments are patched together in the changeover interval. It can be seen from figure 3 that the transition time (2.20) well represents the changeover region and delineates the change from accelerating collapse in the 'initial stage' to decelerating collapse in the 'principal stage'.

3. Discussion

It has been shown that a model based on elliptic deformation and a steady balance between horizontal inertia and buoyancy describes well the collapse of a circular well-mixed region in a stratified fluid for a major part of the time of interest. A characteristic internal Froude number based on the lateral spreading velocity, maximum thickness of the mixed region and ambient buoyancy frequency takes a value near $\frac{5}{8}$ for Wu's (1969) results.

Simple as the model presented is, it compares very favourably with complex 'exact' numerical models in describing the collapsing region (compare figures 2 and 3 with Young & Hirt 1972).

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